

Regular G_δ -diagonals and some upper bounds for cardinality of topological spaces

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Abstract

We prove that, under CH, any space with a regular G_δ -diagonal and caliber ω_1 is separable; a corollary of this result answers, under CH, a question of Buzyakova. For any Urysohn space X , we establish the inequality $|X| \leq wL(X)^{s\Delta_2(X) \cdot \text{dot}(X)}$ which represents a generalization of a theorem of Basile, Bella, and Ridderbos. We also show that if X is a Hausdorff space, then $|X| \leq (\pi\chi(X) \cdot d(X))^{\text{ot}(X) \cdot \psi_c(X)}$; this result implies Šapirovskiĭ's inequality $|X| \leq \pi\chi(X)^{c(X) \cdot \psi(X)}$ which only holds for regular spaces. It is also proved that $|X| \leq \pi\chi(X)^{\text{ot}(X) \cdot \psi_c(X) \cdot aL_c(X)}$ for any Hausdorff space X ; this gives one more generalization of the famous Arhangel'skii's inequality $|X| \leq 2^{\chi(X) \cdot L(X)}$.

1 Introduction

If a space has a G_δ -diagonal, then there are notable restrictions on its cardinal characteristics. For example, if X is a regular Lindelöf space with a G_δ -diagonal, then it has a weaker second countable topology. It is a result of Ginsburg and Woods [9], that $|X| \leq 2^{e(X) \cdot \Delta(X)}$ for any T_1 -space X and hence $|X| \leq \mathfrak{c}$ whenever a T_1 -space X with a G_δ -diagonal has countable extent. However, there are Tychonoff spaces X of arbitrarily large cardinality such that $c(X) \cdot \Delta(X) = \omega$ (see [19]). In particular, having G_δ -diagonal and weak Lindelöf property does not restrict the cardinality of a Tychonoff space.

The situation changes drastically if we assume that a space X has a *regular G_δ -diagonal*, that is, there exists a countable family \mathcal{U} of open neighborhoods of its diagonal $\Delta_X = \{(x, x) : x \in X\}$ in the space $X \times X$ such that $\Delta_X = \bigcap \{\overline{U} : U \in \mathcal{U}\}$. Buzyakova proved in [5] that $|X| \leq \mathfrak{c}$ whenever X is a space with a regular G_δ -diagonal such that $c(X) \leq \omega$. Gotchev extended the result of Buzyakova establishing in [8] that $|X| \leq 2^{c(X) \cdot \overline{\Delta}(X)}$ for any Urysohn space X . It is worth mentioning that every space with a regular G_δ -diagonal is Urysohn.

We recall that a space X has a *zero-set diagonal* if there is a continuous function $f: X \times X \rightarrow \mathbb{R}$ such that $\Delta_X = f^{-1}(0)$. Clearly, having a zero-set diagonal is an even stronger property than having a regular G_δ -diagonal. In [4] Buzyakova established that countable extent of $X \times X$ together with a zero-set diagonal of X imply that X is submetrizable and asked whether any space with a zero-set diagonal and caliber ω_1 also must be submetrizable. In this paper we prove that the answer to this question is positive under the Continuum Hypothesis (see Corollary 3.2).

In [2] Basile, Bella and Ridderbos proved that $|X| \leq wL(X)^{\pi\chi(X)}$ if X is a space with a strong rank 2-diagonal. Here, in Theorem 4.3, we generalize their result by showing that the inequality $|X| \leq wL(X)^{s\Delta_2(X) \cdot \text{dot}(X)}$ is true for any Urysohn space X .

In Corollary 5.9 we show that the inequality $|X| \leq (\pi\chi(X) \cdot d(X))^{\text{ot}(X) \cdot \psi_c(X)}$ is true whenever X is a Hausdorff space. This result implies immediately that for any Hausdorff space X we have $|X| \leq \pi w(X)^{\text{ot}(X) \cdot \psi_c(X)}$. Together with Charlesworth's inequality $d(X) \leq \pi\chi(X)^{c(X)}$ which is valid for regular spaces, our result also implies Šapirovskiĭ's inequality $|X| \leq \pi\chi(X)^{c(X) \cdot \psi(X)}$ which

is known to be true for any regular space X . We also generalize a result of Willard and Dissanayake; they proved in [22] that $|X| \leq 2^{t(X) \cdot \psi_c(X) \cdot \pi\chi(X) \cdot aL_c(X)}$ for any Hausdorff space X . We strengthen their inequality by proving in Theorem 5.1 that the same separation axiom in X guarantees that $|X| \leq \pi\chi(X)^{\text{ot}(X) \cdot \psi_c(X) \cdot aL_c(X)}$. It is worth mentioning that the theorem of Willard and Dissanayake generalizes the famous theorem of Arhangel'skii which states that $|X| \leq 2^{\chi(X) \cdot L(X)}$ whenever X is a Hausdorff space.

2 Notation and terminology

Throughout this paper ω is (the cardinality of) the set of all non-negative integers, ξ , η and α are ordinals and κ , τ , μ and ν are infinite cardinals. The cardinality of the set X is denoted by $|X|$ and $\Delta_X = \{(x, x) \in X^2 : x \in X\}$ is the *diagonal* of X . If \mathcal{U} is a family of subsets of X , $x \in X$, and $G \subset X$ then $\text{st}(G, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap G \neq \emptyset\}$. When $G = \{x\}$ we write $\text{st}(x, \mathcal{U})$ instead of $\text{st}(\{x\}, \mathcal{U})$. If $n \in \omega$, then $\text{st}^n(G, \mathcal{U}) = \text{st}(\text{st}^{n-1}(G, \mathcal{U}), \mathcal{U})$ and $\text{st}^0(G, \mathcal{U}) = G$.

All spaces are assumed to be topological T_1 -spaces. For a subset U of a space X the closure of U in X is denoted by \overline{U} . As usual, $\chi(X)$ and $\psi(X)$ denote respectively the character and the pseudocharacter of X . The *closed pseudocharacter* $\psi_c(X)$ (defined only for Hausdorff spaces X) is the smallest infinite cardinal κ such that for each $x \in X$, there is a collection $\{V(\eta, x) : \eta < \kappa\}$ of open neighborhoods of x such that $\bigcap_{\eta < \kappa} \overline{V(\eta, x)} = \{x\}$. A π -base for X is a collection \mathcal{V} of non-empty open sets in X such that if U is any non-empty open set in X then there exists $V \in \mathcal{V}$ such that $V \subset U$. The π -weight of X is $\pi w(X) = \min\{|\mathcal{V}| : \mathcal{V} \text{ is a } \pi\text{-base for } X\} + \omega$. A family \mathcal{V} of non-empty open sets in X is a local π -base at a point $x \in X$ if for every open neighborhood U of x there is $V \in \mathcal{V}$ such that $V \subset U$. The minimal infinite cardinal κ such that for each $x \in X$ there is a collection $\{V(\eta, x) : \eta < \kappa\}$ of non-empty open subsets of X which is a local π -base for x is called the π -character of X and is denoted by $\pi\chi(X)$.

The *Lindelöf number* of X is $L(X) = \min\{\kappa : \text{every open cover of } X \text{ has a subcover of cardinality } \leq \kappa\} + \omega$. The *weak Lindelöf number* of X , denoted by $wL(X)$, is the smallest infinite cardinal κ such that every open cover of X has a subcollection of cardinality $\leq \kappa$ whose union is dense in X . If $wL(X) = \omega$ then X is called *weakly Lindelöf*. The *weak Lindelöf degree of X with respect to closed sets* is denoted by $wL_c(X)$ and is defined as the smallest infinite cardinal κ such that for every closed subset F of X and every collection \mathcal{V} of open sets in X that covers F , there is a subcollection \mathcal{V}_0 of \mathcal{V} such that $|\mathcal{V}_0| \leq \kappa$ and $F \subset \overline{\bigcup \mathcal{V}_0}$. The *almost Lindelöf number of X with respect to closed sets* is denoted by $aL_c(X)$ and is the smallest infinite cardinal κ such that for every closed subset F of X and every collection \mathcal{V} of open sets in X that covers F , there is a subcollection \mathcal{V}_0 of \mathcal{V} such that $|\mathcal{V}_0| \leq \kappa$ and $\{\overline{V} : V \in \mathcal{V}_0\}$ covers F . A pairwise disjoint collection of non-empty open sets in X is called a *cellular family*. The *cellularity* of X is the cardinal $c(X) = \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a cellular family in } X\} + \omega$. We say that the *o -tightness* of a space X does not exceed κ , or $\text{ot}(X) \leq \kappa$, if for every

family \mathcal{U} of open subsets of X and for every point $x \in X$ with $x \in \bigcup \mathcal{U}$ there exists a subfamily $\mathcal{V} \subset \mathcal{U}$ such that $|\mathcal{V}| \leq \kappa$ and $x \in \bigcup \mathcal{V}$. The *tightness* at $x \in X$ is $t(x, X) = \min\{\kappa : \text{for every } Y \subseteq X \text{ with } x \in \overline{Y}, \text{ there is } A \subset Y \text{ with } |A| \leq \kappa \text{ and } x \in \overline{A}\}$ and the *tightness* of X is $t(X) = \sup\{t(x, X) : x \in X\} + \omega$.

A space X has a G_κ -diagonal if there is a family $\{U_\alpha : \alpha < \kappa\}$ of open sets in $X \times X$ such that $\Delta_X = \bigcap_{\alpha < \kappa} U_\alpha$; if, additionally, $\Delta_X = \bigcap_{\alpha < \kappa} \overline{U_\alpha}$ then X has a *regular G_κ -diagonal*. Clearly, when $\kappa = \omega$ then X has a G_δ -diagonal (respectively, regular G_δ -diagonal). The *diagonal degree* of X , denoted $\Delta(X)$, is the smallest infinite cardinal κ such that X has a G_κ -diagonal (hence $\Delta(X) = \omega$ if and only if X has a G_δ -diagonal). It is worth noting that a space X has a regular G_κ -diagonal for some cardinal κ if and only if X is a Urysohn space. For a Urysohn space X , the minimal cardinal κ such that X has a regular G_κ -diagonal is denoted by $\overline{\Delta}(X)$ and is called the *regular diagonal degree* of X . Given a space X and $n \in \mathbb{N}$, let $s\Delta_n(X)$ be the minimal cardinal κ for which there exists a family $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ of open covers of X such that for any distinct points $x, y \in X$ we have $y \notin \overline{\text{st}^n(x, \mathcal{U}_\alpha)}$ for some $\alpha < \kappa$. A space X has *strong rank n -diagonal* if $s\Delta_n(X) = \omega$.

Condensations are one-to-one and onto continuous mappings. A space X is *submetrizable* if it condenses onto a metrizable space, or equivalently, (X, τ) is *submetrizable* if there exists a topology τ' on X such that $\tau' \subset \tau$ and (X, τ') is metrizable.

For definitions not given here and more information we refer the reader to [7], [8], [10], [12], [14] and [20].

3 On a question of Buzyakova

Here we will establish that caliber ω_1 together with regular G_δ -diagonal is equivalent to separability under the Continuum Hypothesis. We recall that ω_1 is said to be a *caliber* of a space X if any uncountable family \mathcal{U} of non-empty open subsets of X has an uncountable subfamily \mathcal{U}' such that $\bigcap \mathcal{U}' \neq \emptyset$. It is easy to see that every separable space has caliber ω_1 .

Buzyakova proved in [4, Theorem 2.4] that a separable space with a regular G_δ -diagonal condenses onto a second countable Hausdorff space. She also asked (see [4, Question 1.3]) whether zero-set diagonal of X together with ω_1 caliber imply that X is submetrizable. We give a positive answer to this question under the Continuum Hypothesis; the same method gives a generalization, under CH, of Theorem 2.4 of the paper [4].

Theorem 3.1. *Under the Continuum Hypothesis, if X is a space with a regular G_δ -diagonal and caliber ω_1 then X is separable.*

Proof. Since X is a space with caliber ω_1 , we have $c(X) = \omega$. Buzyakova proved in [5] that if a space X has a countable cellularity and a regular G_δ -diagonal then $|X| \leq 2^\omega$, so we have $|X| \leq 2^\omega = \omega_1$. Choose an enumeration $\{x_\alpha : \alpha < \omega_1\}$ of the space X and let $X_\alpha = \overline{\{x_\beta : \beta < \alpha\}}$ for all $\alpha < \omega_1$. It is immediate that $U_\alpha = X \setminus X_\alpha$ is an open subset of X for every α and that the family

$\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ is decreasing and point-countable. Since ω_1 is a caliber of X , it is impossible that all elements of \mathcal{U} be non-empty and therefore $X_\alpha = X$ for some $\alpha < \omega_1$, so X is separable. \square

The following corollary answers Question 3.1 of [4] under CH.

Corollary 3.2. *Under the Continuum Hypothesis, if X has a zero-set diagonal and caliber ω_1 then X is submetrizable.*

Proof. Since zero-set diagonal implies regular G_δ -diagonal, we can apply Theorem 3.1 to see that X must be separable, so it is submetrizable by a theorem of Martin (see [15, Theorem 2.1]). \square

The corollary that follows generalizes Theorem 2.4 of [4] under CH.

Corollary 3.3. *Under the Continuum Hypothesis, if X has a regular G_δ -diagonal and caliber ω_1 then X condenses onto a second countable Hausdorff space.*

Proof. Apply Theorem 3.1 to see that X is separable and hence Theorem 2.4 of [4] is applicable to conclude that X condenses onto a second countable Hausdorff space. \square

4 Bounds given by the weak Lindelöf number

In this section we will show that the weak Lindelöf number and dense o-tightness give an upper bound on the cardinality of spaces with strong rank 2-diagonals; observe that such spaces are automatically Urysohn. Basile, Bella and Ridderbos proved in [2] that if X is a space with a strong rank 2-diagonal then $|X| \leq wL(X)^{\pi\chi(X)}$. Theorem 4.3 below generalizes their result. For its statement we need the following definition.

Definition 4.1. *We will say that the dense o-tightness of X does not exceed κ , or $\text{dot}(X) \leq \kappa$, if for every family \mathcal{U} of open subsets of X whose union is dense in X and for every point $x \in X$ there exists a subfamily $\mathcal{V} \subset \mathcal{U}$ such that $|\mathcal{V}| \leq \kappa$ and $x \in \overline{\bigcup \mathcal{V}}$.*

The observation below follows immediately from the definitions.

Lemma 4.2. *The inequalities $\text{dot}(X) \leq \text{ot}(X)$, $\text{dot}(X) \leq \pi\chi(X)$ and $\text{dot}(X) \leq c(X)$ are valid for every space X .*

Theorem 4.3. *For every Urysohn space X we have $|X| \leq wL(X)^{s\Delta_2(X) \cdot \text{dot}(X)}$.*

Proof. Assume that $wL(X) \leq \lambda$, $\text{dot}(X) \leq \mu$ and, for some infinite cardinal κ , let $\{\mathcal{U}_\eta : \eta < \kappa\}$ be a family witnessing the inequality $s\Delta_2(X) \leq \kappa$. For each $\eta < \kappa$, we can fix a family $\mathcal{D}_\eta \subset \mathcal{U}_\eta$ such that $|\mathcal{D}_\eta| \leq \lambda$ and $\overline{\bigcup \mathcal{D}_\eta} = X$. The family $\mathcal{D} = \bigcup \{\mathcal{D}_\eta : \eta < \kappa\}$ has cardinality not exceeding $\lambda \cdot \kappa$.

It follows from $\text{dot}(X) \leq \mu$ that for every $x \in X$ and $\eta < \kappa$ we can find a family $\mathcal{V}'_\eta(x) \subset \mathcal{D}_\eta$ such that $x \in \overline{\bigcup \mathcal{V}'_\eta(x)}$ and $|\mathcal{V}'_\eta(x)| \leq \mu$. The cardinality of the family $\mathcal{V}_\eta(x) = \{V \in \mathcal{V}'_\eta(x) : V \cap \text{st}(x, \mathcal{U}_\eta) \neq \emptyset\}$ does not exceed μ either and $x \in \overline{\bigcup \mathcal{V}_\eta(x)}$ for any $x \in X$ and $\eta < \kappa$.

Letting $F(x)(\eta) = \mathcal{V}_\eta(x)$ for any point $x \in X$ and any ordinal $\eta < \kappa$ we define a map $F: X \rightarrow [\mathcal{D}]^{\leq \mu}$. To see that F is injective take any pair x, y of distinct points of X . There exists $\eta < \kappa$ such that $y \notin \overline{\text{st}^2(x, \mathcal{U}_\eta)}$. Then $F(x)(\eta) = \mathcal{V}_\eta(x)$ which shows that $x \in \overline{\bigcup \mathcal{V}_\eta(x)}$ and $\bigcup \mathcal{V}_\eta(x) \subset \text{st}^2(x, \mathcal{U}_\eta)$ so $y \notin \overline{\bigcup \mathcal{V}_\eta(x)}$. Therefore $F(y)(\eta) \neq \mathcal{V}_\eta(x) = F(x)(\eta)$ and hence $F(x) \neq F(y)$. This shows that F is an injective map and consequently, $|X| \leq |[\mathcal{D}]^{\leq \mu}| \leq (\lambda \cdot \kappa)^\mu \leq (\lambda \cdot \lambda^\kappa)^\mu \leq \lambda^{\kappa \cdot \mu}$ as promised. \square

Corollary 4.4. *If X is a space with a strong rank 2-diagonal, then $|X| \leq wL(X)^{\text{dot}(X)}$.*

Corollary 4.5. *For every space X of countable dense σ -tightness and strong rank 2-diagonal we have $|X| \leq wL(X)^\omega$.*

Corollary 4.6. *If X is a weakly Lindelöf space with a strong rank 2-diagonal, then $|X| \leq 2^{\text{dot}(X)}$.*

Corollary 4.7. *Assume that X is a weakly Lindelöf space of countable dense σ -tightness and strong rank 2-diagonal. Then $|X| \leq 2^\omega$.*

Corollary 4.8. *If X is a Urysohn space, then*

$$|X| \leq wL(X)^{\pi\chi(X) \cdot s\Delta_2(X)}.$$

Corollary 4.9. *For every Urysohn space X we have*

$$|X| \leq 2^{c(X) \cdot s\Delta_2(X)}.$$

Gotchev established in [8] that $|X| \leq wL(X)^{\chi(X) \cdot \overline{\Delta}(X)}$ for any Urysohn space X so it would be interesting to see whether it is possible to prove the following simultaneous generalization of his result and Corollary 4.8.

Question 4.10. *Is it true that the inequality $|X| \leq wL(X)^{\pi\chi(X) \cdot \overline{\Delta}(X)}$ holds for all Urysohn spaces?*

We will show next that Theorem 4.3 indeed improves the result of Basile, Bella and Ridderbos mentioned in the beginning of this Section.

Example 4.11. It is well known that the Cantor cube $K = \{0, 1\}^{\mathfrak{c}}$ is separable; let X be a countable dense subspace of K . The space X being submetrizable and countable, we have $s\Delta_2(X) = L(X) = wL(X) = \text{dot}(X) = \omega$ while $\pi\chi(X) = \pi\chi(K) = \mathfrak{c}$. Therefore the formula

$$wL(X)^{s\Delta_2(X) \cdot \text{dot}(X)} = \omega^\omega = \mathfrak{c} < 2^\mathfrak{c} = wL(X)^{s\Delta_2(X) \cdot \pi\chi(X)}$$

witnesses that Theorem 4.3 is strictly stronger than the result of Basile, Bella and Ridderbos.

The following example shows that there is even a compact space X such that $\text{dot}(X) < \min\{\text{ot}(X), \pi\chi(X)\}$.

Example 4.12. For every infinite cardinal τ , there exists a compact Hausdorff space X such that $\text{dot}(X) = \tau < \min\{\text{ot}(X), \pi\chi(X)\}$.

Proof. Let $Y = D^\kappa$ be the Cantor cube, where $\kappa = \tau^+$. Denote by Z the space $\kappa + 1$ of ordinal numbers less than or equal to κ endowed with the order topology.

Fix a point $y_0 \in Y$ and let X be the quotient space of the topological sum $Y \oplus Z$ obtained by identifying y_0 with the point $\kappa \in Z$. It is clear that Y , Z , and X are compact Hausdorff spaces. The space X contains closed copies of Y and Z , where $Z' = Z \setminus \{\kappa\}$ accumulates at the point y_0 “outside” of Y . We denote the point $\{y_0, \{\kappa\}\}$ of X by x_0 . Notice that $Y \setminus \{y_0\}$ and Z' are disjoint open subsets of X once Y and Z are identified with the corresponding subspaces of X .

Let \mathcal{U} be a family of open subsets of X such that $\bigcup \mathcal{U}$ is dense in X . Then $\mathcal{V} = \{U \cap Y : U \in \mathcal{U}\}$ is a family of open sets in Y whose union is dense in Y . Since Y has countable cellularity, there exists a countable subfamily \mathcal{V}' of \mathcal{V} such that $\bigcup \mathcal{V}'$ is dense in Y . Take a countable subfamily \mathcal{U}' of \mathcal{U} such that $\mathcal{V}' = \{U \cap Y : U \in \mathcal{U}'\}$. Then $Y \subset \overline{\bigcup \mathcal{U}'}$ and, in particular, $x_0 \in \overline{\bigcup \mathcal{U}'}$. If $x \in X \setminus Y$, that is $x \in Z'$, then we use the inequality $\chi(Z') \leq \tau$ to choose a subset $A \subset Z' \cap \bigcup \mathcal{U}$ such that $x \in \overline{A}$ and $|A| \leq \tau$. Then we take a subfamily \mathcal{W} of \mathcal{U} such that $A \subset \bigcup \mathcal{W}$ and $|\mathcal{W}| \leq \tau$. It is clear that $x \in \overline{A} \subset \bigcup \mathcal{W}$. This implies that $\text{dot}(X) \leq \tau$.

Since Z' is open in X and x_0 compactifies Z' , we see that $\text{ot}(X) = \text{ot}(Z) = \kappa$. The same argument along with the equality $\pi\chi(y_0, Y) = \kappa$ imply that $\pi\chi(X) = \pi\chi(Z) = \kappa$. It follows that $\text{dot}(X) = \tau < \kappa = \min\{\text{ot}(X), \pi\chi(X)\}$. \square

Remark 4.13. We do not know whether the difference between $\text{dot}(X)$ and $\min\{\text{ot}(X), \pi\chi(X)\}$ can be arbitrarily large for a compact space X . However, the difference in question for non-compact spaces can be arbitrary large. To see this one can strengthen the topology of the ordinal space $Z = \kappa + 1$ in Example 4.12 by declaring the points of Z' isolated and taking the sets of the form $Z \setminus A$, with $A \subset Z'$ and $|A| \leq \tau$, as basic open neighborhoods of the point $\{\kappa\}$. Let Z^* be the resulting space. Repeating the construction in Example 4.12 applied to the topological sum of Y and Z^* , we obtain a quotient space X^* satisfying $\text{dot}(X^*) = \omega$ and $\min\{\text{ot}(X^*), \pi\chi(X^*)\} = \kappa$.

5 Bounds on cardinality involving o-tightness

In 1969 Arhangel'skii proved that the inequality $|X| \leq 2^{\chi(X) \cdot L(X)}$ is valid for every Hausdorff space X ([1]). In 1972 Šapironovskii improved this result by showing that $|X| \leq 2^{t(X) \cdot \psi(X) \cdot L(X)}$ ([17]). Later Willard and Dissanayake improved Arhangel'skii's inequality by showing that $|X| \leq 2^{t(X) \cdot \psi_c(X) \cdot \pi\chi(X) \cdot aL_c(X)}$ ([22]). Then Bella and Cammaroto noticed that the cardinal function $\pi\chi(X)$ could be omitted and showed that $|X| \leq 2^{t(X) \cdot \psi_c(X) \cdot aL_c(X)}$ ([3]). Their result is also a

generalization of Šapirovskii's inequality. The following theorem gives another strengthening of the theorem of Willard and Dissanayake; therefore it also generalizes Arhangel'skii's inequality. It is worth noting that $\text{ot}(X) \leq t(X)$ and $\text{ot}(X) \leq c(X)$ for any space X .

Theorem 5.1. *If X is a Hausdorff space, then*

$$(1) \quad |X| \leq \pi\chi(X)^{\text{ot}(X) \cdot \psi_c(X) \cdot aL_c(X)}.$$

Proof. Let $\pi\chi(X) = \tau$ and $\text{ot}(X) \cdot \psi_c(X) \cdot aL_c(X) = \kappa$. For each $x \in X$ choose a π -base \mathcal{U}_x at the point x and a family \mathcal{V}_x of open neighborhoods of x such that $|\mathcal{U}_x| \leq \tau$, $|\mathcal{V}_x| \leq \kappa$ and $\bigcap \{\overline{V} : V \in \mathcal{V}_x\} = \{x\}$.

For every subset A of X let $\mathcal{V}_A = \bigcup \{\mathcal{V}_x : x \in A\}$, $\mathcal{U}_A = \bigcup \{\mathcal{U}_x : x \in A\}$ and $\mathcal{U}_A(V) = \{U : U \subset V, U \in \mathcal{U}_x, x \in A \cap V\}$ whenever V is an open subset of X . Let also $\mathfrak{W}_A = \{\mathcal{W} : |\mathcal{W}| \leq \kappa, \mathcal{W} \subset \mathcal{V}_A, X \setminus \bigcup \{\overline{V} : V \in \mathcal{W}\} \neq \emptyset\}$, $\mathfrak{D}_A = \{\mathcal{O} : |\mathcal{O}| \leq \kappa, \mathcal{O} \subset \mathcal{U}_A\}$ and $\mathfrak{P}_A = \{\mathcal{P} : |\mathcal{P}| \leq \kappa, \mathcal{P} \subset \mathfrak{D}_A, (\bigcap \{\overline{\mathcal{O}} : \mathcal{O} \in \mathcal{P}\}) \setminus A \neq \emptyset\}$. For each $\mathcal{W} \in \mathfrak{W}_A$ we pick a point $p_A(\mathcal{W}) \in X \setminus \bigcup \{\overline{V} : V \in \mathcal{W}\}$ and for each $\mathcal{P} \in \mathfrak{P}_A$ we pick a point $q_A(\mathcal{P}) \in (\bigcap \{\overline{\mathcal{O}} : \mathcal{O} \in \mathcal{P}\}) \setminus A$.

Now let $F_0 = \{z\}$ where z is an arbitrary point in X . Recursively we construct a family $\{F_\eta : \eta < \kappa^+\}$ of subsets of X as follows:

- (i) If $\eta < \kappa^+$ is a limit ordinal then $F_\eta = \bigcup \{F_\xi : \xi < \eta\}$;
- (ii) If $\eta = \xi + 1$ then $F_\eta = F_\xi \cup \{p_{F_\xi}(\mathcal{W}) : \mathcal{W} \in \mathfrak{W}_{F_\xi}\} \cup \{q_{F_\xi}(\mathcal{P}) : \mathcal{P} \in \mathfrak{P}_{F_\xi}\}$.

Observe first that $|F_0| \leq \tau^\kappa$; we will prove by transfinite induction that $|F_\eta| \leq \tau^\kappa$ for each $\eta < \kappa^+$. Assume that for each $\xi < \eta$, where $\eta < \kappa^+$, we have $|F_\xi| \leq \tau^\kappa$. If η is a limit ordinal then clearly $|\bigcup \{F_\xi : \xi < \eta\}| \leq \tau^\kappa$. Now let $\eta = \alpha + 1$. Since $|F_\alpha| \leq \tau^\kappa$, we have $|\{p_{F_\alpha}(\mathcal{W}) : \mathcal{W} \in \mathfrak{W}_{F_\alpha}\}| \leq (\kappa \cdot \tau^\kappa)^\kappa = \tau^\kappa$ and $|\{q_{F_\alpha}(\mathcal{P}) : \mathcal{P} \in \mathfrak{P}_{F_\alpha}\}| \leq ((\tau \cdot \tau^\kappa)^\kappa)^\kappa = \tau^\kappa$. Then it follows from (ii) that $|F_\eta| \leq \tau^\kappa$.

It is clear that the set $F = \bigcup \{F_\eta : \eta < \kappa^+\}$ has cardinality not exceeding τ^κ . We will first prove that F is a closed set in X ; this fact will be used afterwards to show that $F = X$.

Suppose that F is not closed. Then there is $x \in \overline{F} \setminus F$. Let $V \in \mathcal{V}_x$. If V' is any neighborhood of x then $V' \cap V$ is a non-empty open subset of X and therefore there is $y \in F \cap V' \cap V$ and $U \in \mathcal{U}_y$ such that $U \subset V' \cap V$. This shows that for every $V \in \mathcal{V}_x$ we have $x \in \overline{\bigcup \mathcal{U}_F(V)}$. Therefore there exists $\mathcal{O}_V \subset \mathcal{U}_F(V)$ such that $|\mathcal{O}_V| \leq \kappa$ and $x \in \overline{\bigcup \mathcal{O}_V}$. Since $\bigcup \mathcal{O}_V \subset \bigcup \mathcal{U}_F(V) \subset V$, we have $\overline{\bigcup \mathcal{O}_V} \subset \overline{V}$. Therefore $\{x\} = \bigcap \{\overline{\bigcup \mathcal{O}_V} : V \in \mathcal{V}_x\}$. For any $V \in \mathcal{V}_x$ and $U \in \mathcal{O}_V$ choose a point $y = y(V, U) \in F$ such that $U \in \mathcal{U}_y$. It is clear that the cardinality of the set $D = \{y(V, U) : V \in \mathcal{V}_x \text{ and } U \in \mathcal{O}_V\}$ does not exceed κ . Therefore there is $\xi < \kappa^+$ such that $D \subset F_\xi$. Hence, $\mathcal{O}_V \in \mathfrak{D}_{F_\xi}$ whenever $V \in \mathcal{V}_x$. If $\mathcal{P} = \{\mathcal{O}_V : V \in \mathcal{V}_x\}$ then $\mathcal{P} \in \mathfrak{P}_{F_\xi}$ and clearly $x = q_{F_\xi}(\mathcal{P})$. Therefore $x \in F_{\xi+1} \subset F$, which is a contradiction. The proof that F is closed is complete.

Now suppose that there is $x \in X \setminus F$. For each $y \in F$ let $V_y \in \mathcal{V}_y$ be such that $x \notin \overline{V_y}$. Then $\mathcal{V} = \{V_y : y \in F\}$ is an open cover of F . Thus, there exists $\mathcal{V}' \subset \mathcal{V}$

such that $|\mathcal{V}'| \leq \kappa$ and $F \subset \bigcup \{\overline{V_y} : V_y \in \mathcal{V}'\}$. Clearly $x \notin \bigcup \{\overline{V_y} : V_y \in \mathcal{V}'\}$. Let $C = \{y : V_y \in \mathcal{V}'\}$. Then $|C| \leq \kappa$ and therefore there is $\xi < \kappa^+$ such that $C \subset F_\xi$. Since $X \setminus \bigcup \{\overline{V_y} : V_y \in \mathcal{V}'\} \neq \emptyset$, we have $p_{F_\xi}(\mathcal{V}') \notin \bigcup \{\overline{V_y} : V_y \in \mathcal{V}'\}$ and at the same time $p_{F_\xi}(\mathcal{V}') \in F_{\xi+1} \subset F \subset \bigcup \{\overline{V_y} : V_y \in \mathcal{V}'\}$. This contradiction completes the proof. \square

Corollary 5.2. *For every Hausdorff space X we have*

$$|X| \leq \pi\chi(X)^{\text{ot}(X) \cdot \psi(X) \cdot L(X)}.$$

Proof. The claim follows directly from Theorem 5.1 and the fact that if X is a Hausdorff space then $\psi_c(X) \leq \psi(X) \cdot L(X)$ (see [14, 2.9(c)]). \square

Corollary 5.3. *If X is a Urysohn space, then*

$$(2) \quad |X| \leq \pi\chi(X)^{\text{ot}(X) \cdot \psi(X) \cdot aL_c(X)}.$$

Proof. The inequality follows directly from Theorem 5.1 and the fact that if X is a Urysohn space then $\psi_c(X) \leq \psi(X) \cdot aL_c(X)$ (see [13, Lemma 2.1]). \square

Corollary 5.4. *If X is a Hausdorff space with $\pi\chi(X) \leq 2^{\text{ot}(X) \cdot \psi_c(X) \cdot aL_c(X)}$ then $|X| \leq 2^{\text{ot}(X) \cdot \psi_c(X) \cdot aL_c(X)}$.*

We present next an example which shows that Theorem 5.1 indeed improves the theorem of Willard and Dissanayake. Recall that for a Tychonoff space X , the expression $C_p(X)$ stands for the set of all real-valued continuous functions on X endowed with the pointwise convergence topology. For the basic facts about the spaces $C_p(X)$ we refer the reader to the book [21].

Example 5.5. Apply Theorem 2.1 of [16] to see that there exists a Tychonoff space Z with the following properties:

- (i) $Z = D \cup A \cup \{p\}$ where D is a countable dense set of isolated points of Z while the sets A, D and $\{p\}$ are disjoint;
- (ii) the subspace $A \cup \{p\}$ is compact and p is its unique non-isolated point;
- (iii) $|A| = \mathfrak{c}$ and the space $C_p(Z)$ is Lindelöf.

Observe first that the space Z is separable and therefore $C_p(Z)$ has a weaker second countable topology and, in particular, $\psi(C_p(Z)) = \omega$ (see [21, Problem 173]). Take a set $A_0 \subset A$ such that $|A_0| = |A \setminus A_0| = \mathfrak{c}$ and let $u \in \mathbb{R}^Z$ be the function such that $u(z) = 0$ for all $z \in A_0$ and $u(z) = 1$ whenever $z \in Z \setminus A_0$; it is clear that $u \in \mathbb{R}^Z \setminus C_p(Z)$. We will prove even more, namely, that

(*) if $Q \subset C_p(Z)$ and $|Q| < \mathfrak{c}$, then $u \notin \overline{Q}$ (the closure is taken in \mathbb{R}^Z).

To verify (*) note first that for every $f \in Q$ there exists a countable set $A_f \subset A$ such that $f(z) = f(p)$ for any $z \in A \setminus A_f$. The cardinality of the set $A' = \bigcup \{A_f : f \in Q\}$ is strictly less than \mathfrak{c} so we can find points $z_0 \in A_0 \setminus A'$ and $z_1 \in (A \setminus A_0) \setminus A'$. It is straightforward that $U = \{g \in \mathbb{R}^Z : |g(z_0)| < \frac{1}{3} \text{ and } |g(z_1) - 1| < \frac{1}{3}\}$ is an open neighborhood of u in \mathbb{R}^Z such that $U \cap Q = \emptyset$; this settles (*).

Take a countable dense subset E in the space \mathbb{R}^Z such that $u \in E$. Then $X = C_p(Z) \cup E$ is a separable Lindelöf subspace of \mathbb{R}^Z and it follows from (*) that $t(X) = \mathfrak{c}$. Besides, $\psi(C_p(Z)) = \omega$ easily implies that $\psi(X) = \omega$. The space X is Tychonoff and Lindelöf so we have $\psi_c(X) = \psi(X) = \omega$ and $aL_c(X) = L(X) = \omega$. Since $X \subset \mathbb{R}^Z$ and $|Z| \leq \mathfrak{c}$, we have $w(X) \leq \mathfrak{c}$. Hence $\pi\chi(X) \leq \mathfrak{c}$. Also, since X is separable, we have $\text{ot}(X) = \omega$. Therefore the formula

$$\pi\chi(X)^{\text{ot}(X) \cdot \psi_c(X) \cdot aL_c(X)} \leq \mathfrak{c}^\omega = \mathfrak{c} < 2^\mathfrak{c} = 2^{t(X) \cdot \psi_c(X) \cdot \pi\chi(X) \cdot aL_c(X)}$$

witnesses that Theorem 5.1 is strictly stronger than the inequality of Willard and Dissanayake.

The result in Theorem 5.1 should be compared with Šapironskii's inequality

$$(3) \quad |X| \leq \pi\chi(X)^{c(X) \cdot \psi(X)}.$$

proved in [18] for regular spaces. If X is the Katětov extension $\kappa\omega$ of the set ω with the discrete topology, then X is Urysohn and $\pi\chi(X) = c(X) = \psi(X) = \omega$ while $|X| = 2^\mathfrak{c}$; this shows that Šapironskii's inequality is not true if we drop the regularity of X . The inequality (2) shows one of the possible ways to find a statement analogous to (3) that holds for Urysohn spaces.

Our next result will allow us to obtain a direct strengthening of (3) for Hausdorff spaces.

Theorem 5.6. *If X is a Hausdorff space and $A \subset X$ then*

$$|\overline{A}| \leq (\pi\chi(X) \cdot |A|)^{\text{ot}(X) \cdot \psi_c(X)}.$$

Proof. Let $\psi_c(X) = \nu$, $\text{ot}(X) = \mu$ and $\pi\chi(X) = \tau$. For each $x \in X$ choose a local π -base \mathcal{U}_x at the point x with $|\mathcal{U}_x| \leq \tau$ and a family \mathcal{V}_x of open neighborhoods of x such that $\{x\} = \bigcap \{\overline{V} : V \in \mathcal{V}_x\}$ and $|\mathcal{V}_x| \leq \nu$; let $\mathcal{U} = \bigcup \{\mathcal{U}_x : x \in A\}$.

For any $x \in \overline{A}$ and $V \in \mathcal{V}_x$ the cardinality of the family $\mathcal{U}_A(V) = \{U : U \subset V, U \in \mathcal{U}_y, y \in A \cap V\} \subset \mathcal{U}$ does not exceed $\tau \cdot |A|$. If V' is any neighborhood of x then $V' \cap V$ is a non-empty open subset of X and therefore there is $y \in A \cap V' \cap V$ and $U \in \mathcal{U}_y$ such that $U \subset V' \cap V$. This shows that for every $V \in \mathcal{V}_x$ we have $x \in \overline{\bigcup \mathcal{U}_A(V)}$. Therefore for every $V \in \mathcal{V}_x$ there exists $\mathcal{O}_V \subset \mathcal{U}_A(V)$ such that $|\mathcal{O}_V| \leq \mu$ and $x \in \overline{\bigcup \mathcal{O}_V}$.

Since $\bigcup \mathcal{O}_V \subset \bigcup \mathcal{U}_A(V) \subset V$, we have the inclusion $\overline{\bigcup \mathcal{O}_V} \subset \overline{V}$. Therefore $\{x\} = \bigcap \{\overline{\bigcup \mathcal{O}_V} : V \in \mathcal{V}_x\}$. Observing that $|\mathcal{U}| \leq \tau \cdot |A|$ and every \mathcal{O}_V is a subfamily of \mathcal{U} of cardinality $\leq \mu$ we conclude that the cardinality of the collection $\mathcal{O} = \{\mathcal{O}_V : V \in \mathcal{V}_x, x \in A\}$ does not exceed $(\tau \cdot |A|)^\mu$. Hence there are at most $((\tau \cdot |A|)^\mu)^\nu$ -many intersections of ν -many elements of \mathcal{O} . We already saw that every point of \overline{A} is an intersection of ν -many elements of \mathcal{O} so $|\overline{A}| \leq (\tau \cdot |A|)^{\mu \cdot \nu}$. \square

Corollary 5.7. *If X is a Hausdorff space, then $|\overline{A}| \leq \pi\chi(X)^{\text{ot}(X) \cdot \psi_c(X)}$ whenever $A \subset X$ and $|A| \leq \pi\chi(X)^{\text{ot}(X) \cdot \psi_c(X)}$.*

Corollary 5.8. *Assume that X is a Hausdorff space and $A \subset X$. Then*

$$|\overline{A}| \leq (\pi\chi(\overline{A}) \cdot |A|)^{\text{ot}(\overline{A}) \cdot \psi_c(\overline{A})}.$$

Corollary 5.9. *For every Hausdorff space X we have*

$$|X| \leq (\pi\chi(X) \cdot d(X))^{\text{ot}(X) \cdot \psi_c(X)}.$$

Observe that Corollary 5.9 implies Šapironskii's inequality (3) because for every space X we have $\text{ot}(X) \leq c(X)$ while $\psi_c(X) = \psi(X)$ and $d(X) \leq \pi\chi(X)^{c(X)}$ whenever X is a regular space (see [6]).

Corollary 5.10. *If X is a Hausdorff space, then*

$$|X| \leq d(X)^{\pi\chi(X) \cdot \text{ot}(X) \cdot \psi_c(X)}.$$

Theorem 5.11. *Let X be a Hausdorff space. Then*

$$|X| \leq \pi w(X)^{\text{ot}(X) \cdot \psi_c(X)}.$$

Proof. It follows directly from Corollary 5.9 and the fact that for every topological space X we have $\pi w(X) = \pi\chi(X) \cdot d(X)$ (see [12, 3.8(b)]). \square

Observation 5.12. It is natural to ask whether the result of Bella and Cammaroto is stronger than our inequality (1). This would happen if $2^{t(X)} \leq \pi\chi(X)^{\text{ot}(X)}$ for any Hausdorff space X . However, this is false even for compact spaces. Indeed, if X is the Tychonoff cube $[0, 1]^{\mathfrak{c}}$, then $\pi\chi(X) = t(X) = \mathfrak{c}$ and $\text{ot}(X) \leq c(X) = \omega$ so $\pi\chi(X)^{\text{ot}(X)} = \mathfrak{c} < 2^{\mathfrak{c}} = 2^{t(X)}$.

If the inequality $\pi\chi(X)^{\text{ot}(X)} \leq 2^{t(X)}$ were true for all Hausdorff spaces, then Theorem 5.1 would imply the inequality of Bella and Cammaroto. However, this inequality does not hold either: Take X to be the Σ -product of $2^{\mathfrak{c}}$ -many real lines and observe that $t(X) = \omega$ while $\pi\chi(X) = 2^{\mathfrak{c}}$ so $2^{t(X)} < \pi\chi(X) = \pi\chi(X)^{\text{ot}(X)}$.

To see that Corollary 5.4 gives new information, it suffices to prove that there exists a Hausdorff space X such that $\pi\chi(X) > 2^{\text{ot}(X) \cdot \psi_c(X) \cdot \text{al}_c(X)}$. We will show that there are models of ZFC in which such a space exists and is even normal.

Theorem 5.13. *There is a model of ZFC in which we can find a regular (and hence normal) hereditarily Lindelöf space X such that $\pi w(X) > \mathfrak{c}$.*

Proof. Hajnal and Juhász proved in [11] that there exists a model of ZFC in which GCH holds and we can find a set E of cardinality ω_1 and a family \mathcal{A} of subsets of E such that $|\mathcal{A}| = \omega_2$ and

- (a) if $k \in \mathbb{N}$ and $\{A_{nm} : n \in \omega, 1 \leq m \leq k\}$ is a subfamily of \mathcal{A} such that $A_{nm} \neq A_{n'm'}$ whenever $(n, m) \neq (n', m')$, then the set $E \setminus \bigcup_{n \in \omega} B_n$ is countable provided that every B_n is the intersection $B_n^1 \cap \dots \cap B_n^k$ in which the set B_n^i is either A_{ni} or $E \setminus A_{ni}$ for each $i \leq k$.

- (b) for any $x \in E$ and countable $B \subset E \setminus \{x\}$, there exists $A \in \mathcal{A}$ such that $x \in A \subset E \setminus B$.

Let Y be the set E with the topology generated by the family $\{A, E \setminus A : A \in \mathcal{A}\}$ as a subbase. It was proved in [11] that Y is a regular zero-dimensional hereditarily Lindelöf space. Denote by \mathcal{C} the family of all open countable subsets of Y ; it is immediate that the set $G = \bigcup \mathcal{C}$ is countable. We claim that the space $X = Y \setminus G$ is as promised; of course, we only need to prove that $\pi w(X) = \omega_2$.

Striving for a contradiction, assume that there is a family \mathcal{B} of non-empty open subsets of X such that $|\mathcal{B}| \leq \omega_1$ and \mathcal{B} is a π -base in X . Observe first that it follows from our choice of X that all elements of \mathcal{B} are uncountable. If infinitely many elements of \mathcal{A} are countable, then we can find a faithfully indexed subfamily $\mathcal{A}' = \{A_n : n \in \omega\}$ of the family \mathcal{A} whose all elements are countable. However, this implies that $E \setminus \bigcup \mathcal{A}'$ is uncountable which is a contradiction with (a). Therefore at most finitely many elements of \mathcal{A} are countable and hence we can find a family $\mathcal{E} \subset \mathcal{A}$ such that $|\mathcal{E}| = \omega_2$ and $A \cap X \neq \emptyset$ for all $A \in \mathcal{E}$.

Observe that $A \cap X$ is a non-empty open subset of X for any $A \in \mathcal{E}$ so it follows from the fact that \mathcal{B} is a π -base in X and $|\mathcal{B}| \leq \omega_1$ that there is $B \in \mathcal{B}$ such that the set $\{A \in \mathcal{E} : B \subset A\}$ has cardinality ω_2 . In particular, we can find a faithfully indexed family $\{A_n : n \in \omega\} \subset \mathcal{E}$ such that $B \subset A_n$ for all $n \in \omega$. As a consequence, $B \cap (E \setminus A_n) = \emptyset$ for each $n \in \omega$ and hence the set $E \setminus \bigcup \{E \setminus A_n : n \in \omega\} \supset B$ is uncountable which is a contradiction with (a). This contradiction proves that $\pi w(X) = \omega_2$. \square

Corollary 5.14. *For the space X from Theorem 5.13 we have $\pi\chi(X) = 2^c > c = 2^{c(X) \cdot \psi(X) \cdot L(X)}$ and therefore $\pi\chi(X) > 2^{\text{ot}(X) \cdot \psi_c(X) \cdot aL_c(X)}$. Thus, Corollary 5.4 gives new information, at least consistently.*

Proof. If $\pi\chi(X) \leq c = \omega_1$, then it follows from $|X| = \omega_1$ that $\pi w(X) \leq \omega_1$ which is a contradiction with Theorem 5.13. Therefore $\pi\chi(X) = \omega_2 = 2^c$ so all that is left is to note that $c(X) = \psi(X) = L(X) = \omega$ because X is hereditarily Lindelöf. Finally, observe that it follows from the regularity of X that $\psi_c(X) = \psi(X)$ and $aL_c(X) \leq L(X)$. \square

The following question seems to be interesting because if it has an affirmative answer, the respective statement will be a simultaneous generalization of Theorem 5.1 and Šapírovskii's inequality (3) in the class of T_3 -spaces.

Question 5.15. *Is the inequality*

$$|X| \leq \pi\chi(X)^{\text{ot}(X) \cdot \psi(X) \cdot wL_c(X)}$$

true for every regular space X ?

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